INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 5, 805–816 (1985).

A NOTE ON THE STABILITY AND ACCURACY OF C^n FINITE ELEMENTS IN STEADY DIFFUSION–CONVECTION PROBLEMS

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SUMMARY

The paper is concerned with stability and accuracy of an nth order Lagrangian family of finite element steady-state solutions of the diffusion–convection equation, and furthermore is concerned with the stability and the accuracy of on mth kind Hermitian family of finite element solutions. We discuss the stability of the numerical solution based on the fact that the characteristic finite element solution can be expressed approximately as a rational function of cell Peclet number $Pe_c(=uh/\kappa)$. Moreover, it is shown that by eliminating derivatives and by using the interpolation method over elements a stable solution is obtained over the domain independent of Pe_c for $P^{1,3}$, and for $P^{2,5}$ the stable solution is obtained for Pe_c less than 44·4.

KEY WORDS Diffusion–Convection Finite Element Method Hermite Interpolation Function Numerical Analysis Stability

1. INTRODUCTION

It is a well known fact that, when the finite element method is applied to diffusion–convection problems, non-physical spatial oscillations occur when the velocity increases.[1,2] In order to avoid these oscillations the upwind finite difference method (UPFDM) or the upwind finite element method (UPFEM)[3] are proposed as numerical methods. Before using UPFEM, we had examined the nth order Lagrangian family of finite elements: $P^{0,n}$ (which belongs to class C^0), so that it is necessary to re-examine the effectiveness and a permitted limit of the standard Galerkin finite element method (SGFEM).[4,5] In this paper, steady diffusion–convection problems are dealt with, and then the stability and the accuracy of the FEM with nth order Lagrange polynomials (nth order SGFEM) and the SGFEM with nth order Hermitian polynomials of the mth kind are discussed. In Section 2 a canonical form of the mth kind nth order FE solution with the Hermite polynomials is given for the 1-dimensional case. In Section 3 the stability of the FEM with the 3rd order Hermite polynomials of the first kind: $P^{1,3}$ (which belong to class C^1) and the 5th order Hermite polynomials of the second kind: $P^{2,5}$ (which belong to class C^2) are discussed. In Section 4, on the basic of these findings, it is shown that the stable region of the FE solution can be expanded by eliminating the internal nodal value[6] provided that for the Lagrangian family of finite elements, the exponential function is used as the finite element interpolation. On the other hand in the case of the Hermitian family of finite elements it is shown that the stable region can be expanded by eliminating the derivative at the node.

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0271–2091/85/090805–12$01.20

Received 7 March 1985

2. BASIC EQUATIONS

The 1-dimensional model under consideration is

$$-\kappa\,\mathrm{d}^2\phi(x)/\mathrm{d}x^2 + u\,\mathrm{d}\phi(x)/\mathrm{d}x = q, \quad \text{in } x \in \Omega = (a, b) \tag{1}$$

$$\phi(a) = c, \quad \phi(b) = d, \tag{2}$$

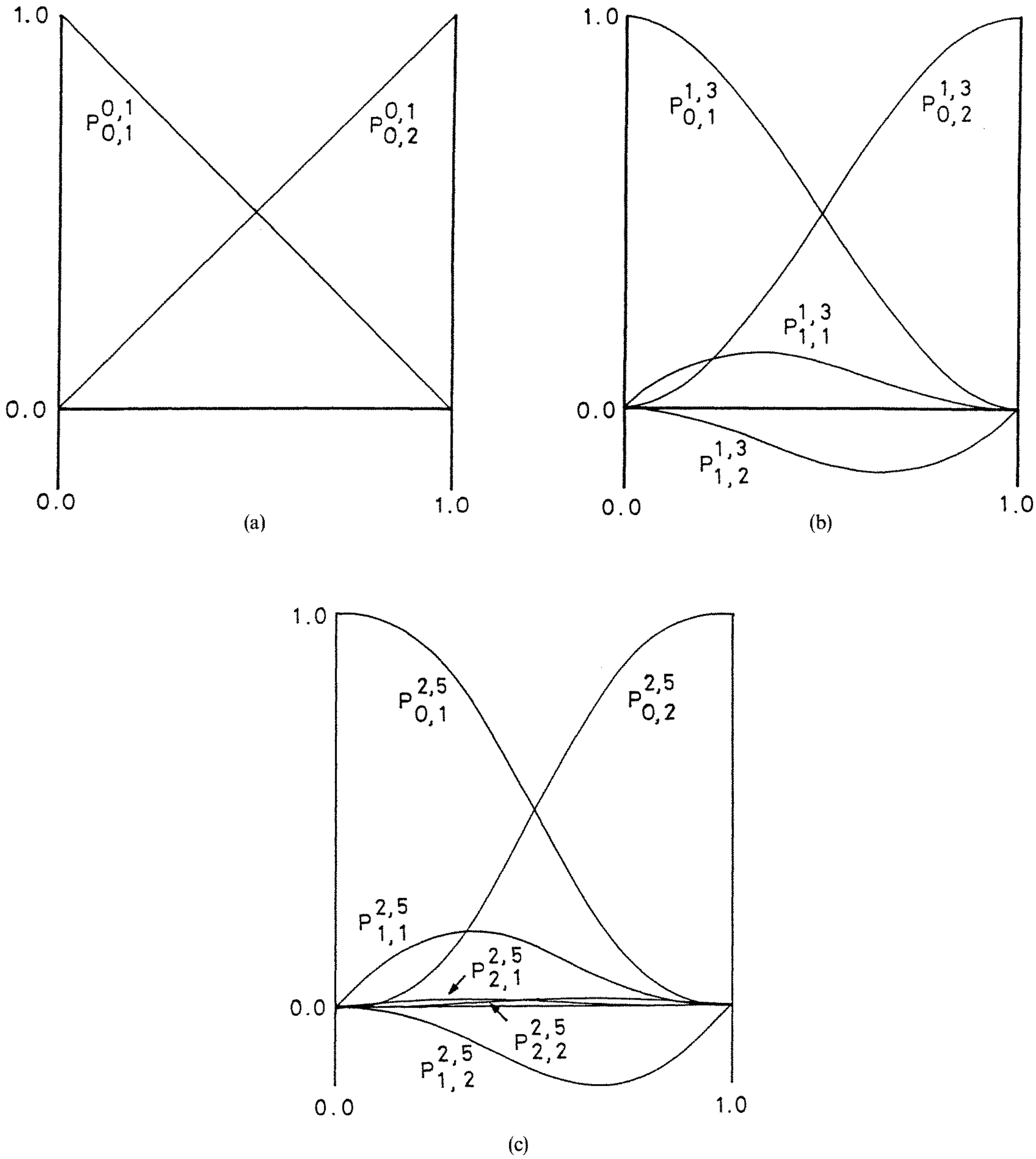

Figure 1. Shape functions of a Hermitian family: (a) zeroth kind ($m = 0$) (Lagrangian family); (b) first kind ($m = 1$); (c) second kind ($m = 2$)

where $\kappa(>0), u(>0)$ and q are constants. The general solution of (1) is given by

$$\phi(x) = A\exp(ux/\kappa) + B + Qx, \tag{3}$$

where A and B are arbitrary constants determined by equation (2) and $Q = q/u$.

The weighted-residual weak form for (1) is

$$\int_\Omega (\mathrm{d}W(x)/\mathrm{d}x)(\kappa\,\mathrm{d}\phi(x)/\mathrm{d}x)\,\mathrm{d}x + \int_\Omega W(x)(u\,\mathrm{d}\phi(x)/\mathrm{d}x)\,\mathrm{d}x = \int_\Omega W(x)q\,\mathrm{d}x. \tag{4a}$$

Here $W(x)$ is the weighting function, satisfying $W(x) = 0$ on Γ.

At each element the local interpolation function $\phi^e(x)$ and the local weighting function $W^e(x)$ are represented by the following matrix forms:

$$\phi^e(x) = \{N\}^{\mathrm{T}}\{\tilde{\phi}\}, \tag{4b}$$

$$W^e(x) = \{N\}^{\mathrm{T}}. \tag{4c}$$

For example for $P^{2,5}$, $\{N\}$ and $\{\tilde{\phi}\}$ take the following forms:

$$\{\tilde{\phi}\}^{\mathrm{T}} = \{\phi_i, \phi'_i, \phi''_i, \phi_j, \phi'_j, \phi''_j\}, \tag{5a}$$

$$\{N\}^{\mathrm{T}} = \{P^{2,5}_{0,i}(x), P^{2,5}_{1,i}(x), P^{2,5}_{2,i}(x), P^{2,5}_{0,j}(x), P^{2,5}_{1,j}(x), P^{2,5}_{2,j}(x)\}, \tag{5b}$$

where the superscript $'$ denotes differentiation.

Schematically these shape functions of the Hermitian family are shown in Figure 1.

For $P^{0,1}$, $P^{1,3}$ and $P^{2,5}$ at node k, the FEM equation with Hermite interpolation polynomials of the mth kind has the form

$$\sum_{l=0}^{m} (C^l_{k-1}\phi^{(l)}_{k-1} + C^l_k\phi^{(l)}_k + C^l_{k+1}\phi^{(l)}_{k+1}) = \tilde{Q}_k, \tag{6}$$

where (l) denotes the lth derivative of the function and C^l_k denotes coefficients which depend on the element length h and the cell Peclet number $Pe_c = uh/\kappa$.

3. CHARACTERISTIC SOLUTION OF THE FEM EQUATION

By adding equations of two adjacent elements and eliminating the derivative term of (ϕ), the following canonical form is derived:

$$-a_n\phi^{(0)}_{k-1} + (a_n + b_n)\phi^{(0)}_k - b_n\phi^{(0)}_{k+1} = Q_k. \tag{7}$$

Here the coefficients $a_n = a_n(Pe_c)$, $b_n = b_n(Pe_c)$ are expressed by polynomials in the cell Peclet number. For example, there is a 4th order polynomial for $P^{1,3}$ and an 18th order polynomial for $P^{2,5}$.

These results are listed in Table I. Here the characteristic solutions Λ_1 and Λ_2 are easily obtained as

$$\Lambda_1 = b_n(Pe_c)/a_n(Pe_c), \tag{8a}$$

$$\Lambda_2 = 1. \tag{8b}$$

Therefore, the general solution of equation (7) is expressed as

$$\phi_k = A_h[b_n(Pe_c)/a_n(Pe_c)]^k + B_h + Q_h. \tag{9}$$

Comparing (3) with (9) the characteristic solution $\exp(uh/\kappa)$ is approximated by Λ_1 in which both

Table I. Parameters a_n and b_n

	$a_n(Pe_c)$	$b_n(Pe_c)$
$P^{0,1}$	$1 - Pe_c/2$	$1 + Pe_c/2$
$P^{0,2}$	$1 - Pe_c/2 + Pe_c^2/12$	$1 + Pe_c/2 + Pe_c^2/12$
$P^{0,3}$	$1 - Pe_c/2 + Pe_c^2/10 - Pe_c^3/120$	$1 + Pe_c/2 + Pe_c^2/10 + Pe_c^3/120$
$P^{0,4}$	$1 - Pe_c/2 + 3Pe_c^2/28 - Pe_c^3/84$ $+ Pe_c^4/1680$	$1 + Pe_c/2 + 3Pe_c^2/28 + Pe_c^3/84$ $1 + Pe_c^4/1680$
$P^{1,3}$	$1 - Pe_c/2 + Pe_c^2/9 - 7Pe_c^3/540 + Pe_c^4/900$	$1 + Pe_c/2 + Pe_c^2/9 + 7Pe_c^3/540 + Pe_c^4/900$
$p^{2,5}$	$1 - (Pe_c/2) + 2327(Pe_c/2)^2/2700$ $- 1427(Pe_c/2)^3/2700$ $+ 69{,}931(Pe_c/2)^4/255{,}150$ $- 43{,}771(Pe_c/2)^5/364{,}500$ $+ 1{,}201{,}669(Pe_c/2)^6/26{,}244{,}000$ $- 80{,}489(Pe_c/2)^7/5{,}248{,}800$ $+ 109{,}050{,}511(Pe_c/2)^8/23{,}808{,}556{,}800$ $- 820{,}877(Pe_c/2)^9/680{,}244{,}480$ $+ 176{,}963(Pe_c/2)^{10}/29{,}386{,}561{,}536$ $- 13{,}009{,}769(Pe_c/2)^{11}/11{,}108{,}120{,}260{,}608$ $+ 298{,}265(Pe_c/2)^{12}/1{,}586{,}874{,}322{,}944$ $- 2{,}939{,}975(Pe_c/2)^{13}/99{,}973{,}082{,}345{,}472$ $+ 24{,}565(Pe_c/2)^{14}/7{,}140{,}934{,}453{,}248$ $- 250{,}895(Pe_c/2)^{15}/599{,}838{,}494{,}072{,}832$ $+ 2675(Pe_c/2)^{16}/85{,}691{,}213{,}438{,}976$ $- 1325(Pe_c/2)^{17}/514{,}147{,}280{,}633{,}856$ $+ 125(Pe_c/2)^{18}/1{,}542{,}441{,}841{,}901{,}568$	$1 + (Pe_c/2) + 2327(Pe_c/2)^2/2700$ $+ 1427(Pe_c/2)^3/2700$ $+ 69{,}931(Pe_c/2)^4/255{,}150$ $+ 43{,}771(Pe_c/2)^5/364{,}500$ $+ 1{,}201{,}669(Pe_c/2)^6/26{,}244{,}000$ $+ 80{,}489(Pe_c/2)^7/5{,}248{,}800$ $+ 109{,}050{,}511(Pe_c/2)^8/23{,}808{,}556{,}800$ $+ 820{,}877(Pe_c/2)^9/680{,}244{,}480$ $+ 176{,}963(Pe_c/2)^{10}/29{,}386{,}561{,}536$ $+ 13{,}009{,}769(Pe_c/2)^{11}/11{,}108{,}120{,}260{,}608$ $+ 298{,}265(Pe_c/2)^{12}/1{,}586{,}874{,}322{,}944$ $+ 2{,}939{,}975(Pe_c/2)^{13}/99{,}973{,}082{,}345{,}472$ $+ 24{,}565(Pe_c/2)^{14}/7{,}140{,}934{,}453{,}248$ $+ 250{,}895(Pe_c/2)^{15}/599{,}838{,}494{,}072{,}832$ $+ 2675(Pe_c/2)^{16}/85{,}691{,}213{,}438{,}976$ $+ 1325(Pe_c/2)^{17}/514{,}147{,}280{,}633{,}856$ $+ 125(Pe_c/2)^{18}/1{,}542{,}441{,}841{,}901{,}568$

$Pe_c = uh/\kappa$

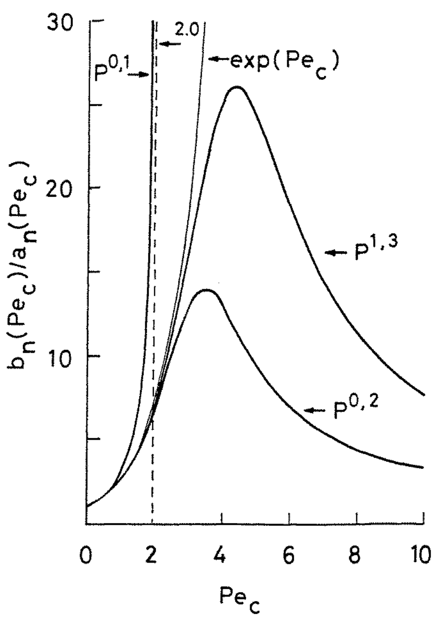

Figure 2. Characteristic solution to equation (7)

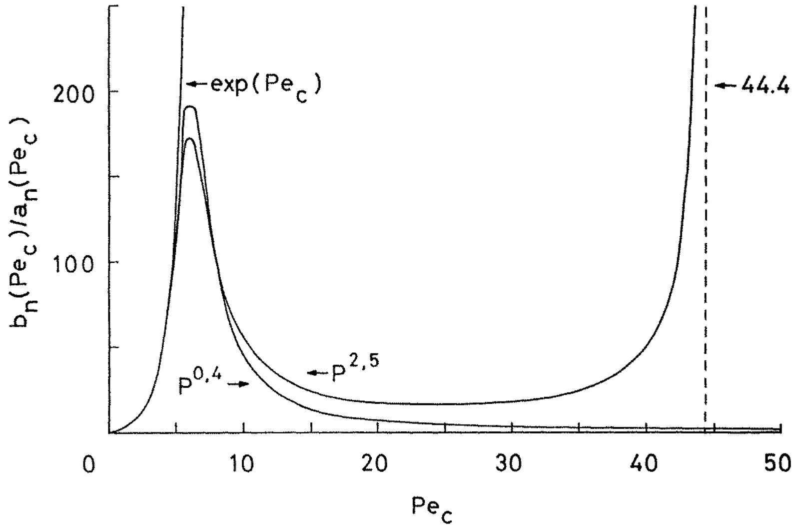

Figure 3. Characteristic solution to equation (7)

numerator and denominator have same order rational function. By examining Λ_1 we find the characteristics of the FE solution with the Hermitian family of finite elements.

Note that in the case of the Lagrangian family of finite elements ($m = 0$) each nodal value is expressed by a rational function (Table I) but for the Hermitian family of finite elements the nodal value is not completely equivalent to one so that a derivative at each node is associated with the function's value.

Now the stability for the general solution ϕ_k is assured only if $\Lambda_1 = b_n/a_n > 0$, since $\Lambda_2 = 1$. However, the property of Λ_1 varies with n, as shown in Figure 2. In the case of the Hermitian family of finite elements ($P^{1,3}$ in Figure 2), as in the even-order Lagrangian family of finite elements ($P^{0,2}$ in Figure 2), Λ_1 is positive for all $Pe_c(\geqslant 0)$, and ϕ_k is unconditionally stable. However, Λ_1 has an extreme value at $Pe_c = 4{\cdot}367$. Also Λ_1 approaches unity as $Pe_c \rightarrow \infty$, and then the linear independence of the fundamental solutions Λ_1, and Λ_2 is gradually violated. Accordingly a non-physical solution appears. As shown in Figure 3, there exists an extreme value for $P_{2,5}$ at $Pe_c = 6{\cdot}1178$, and at $Pe_c = 44{\cdot}4\,\Lambda_1$ diverges to $+\infty$, and then Λ_1 has a negative value when the value of Pe_c exceeds 44·4. Accordingly the stable solution is not obtained. In the above discussion the Dirichlet boundary condition (2) plays an important role. If we use the Neumann condition $\phi'(b) = Q$ instead of $\phi(b) = d$, $A = A_h$ is equal to 0, and then the oscillation does not occur. It is considered that the actual physical model exists between these two conditions.

4. NUMERICAL RESULTS

We show numerical results for equation (1) with $\phi(0) = 1$ and $\phi(1) = 0$, where the element length $h = 0{\cdot}1$, the number of elements $N = 10$ and $\kappa = 1$.

We now discuss ϕ_k at node k. As shown in Figure 4 the non-physical oscillation is obtained as the cell Peclet number is increased for $P^{2,5}$. But for $P^{1,3}$ the oscillation is not obtained. As shown in Figure 5 when the cell Peclet number increases, the value of the FE solution decreases all over the domain even for $P^{1,3}$ and the non-physical solution is obtained. Namely, at the first node of an entrance the value of the derivative becomes larger than the value at the second node. This result is extremely interesting in that it differs from the result in the case of the

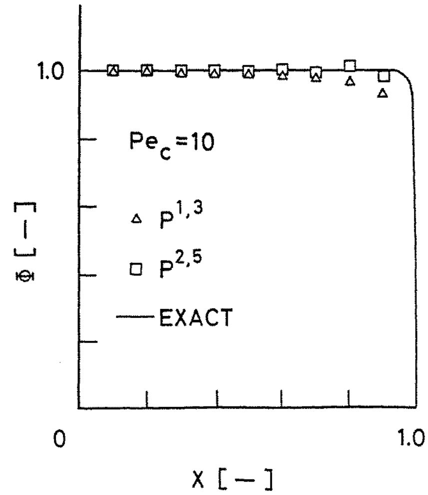

Figure 4. Solution of diffusion–convection equation with Hermitian polynomials $P^{1,3}$, $P^{2,5}$

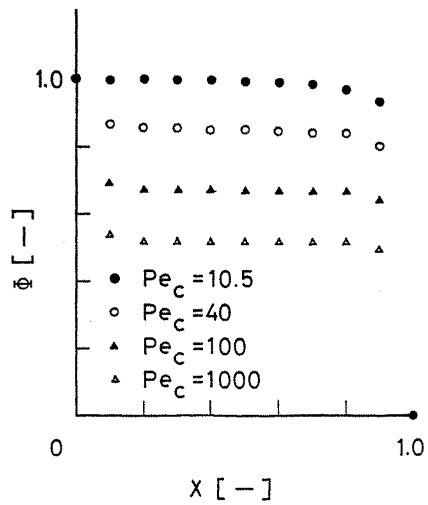

Figure 5. Solution of diffusion–convection equation with Hermitian polynomial $P^{1,3}$

even-order Lagrangian family of finite elements; when the value of Pe_c is great than that of Pe_c in the consistent domain (see that Appendix), the FE solution approaches a solution $\phi(=1-x)$ in the case $u=0$ of equation (1). In the case of the non-oscillation the relative error at the point $x=0{\cdot}9$, which is a conceivable occurrence of the maximum error, is shown in Figure 6. These results can also be understood from the characteristic solution.

Next, we examine derivatives ϕ'_k, ϕ''_k at each node. The errors of derivatives at the exist ($x=1{\cdot}0$) are shown in Figure 7. In the case of $Pe_c=1{\cdot}9$ the gradient error of $P^{0,1}$ is $-48{\cdot}7$ per cent, on the other hand the gradient errors of $P^{1,3}$ and $P^{2,5}$ are $-2{\cdot}78$ per cent and $-0{\cdot}0450$ per cent, respectively: the higher accuracy is obtained for the small cell Peclet number. Now so for as we observe only the derivative at each node for $P^{1,3}$, the oscillations do not occur when Pe_c is less than

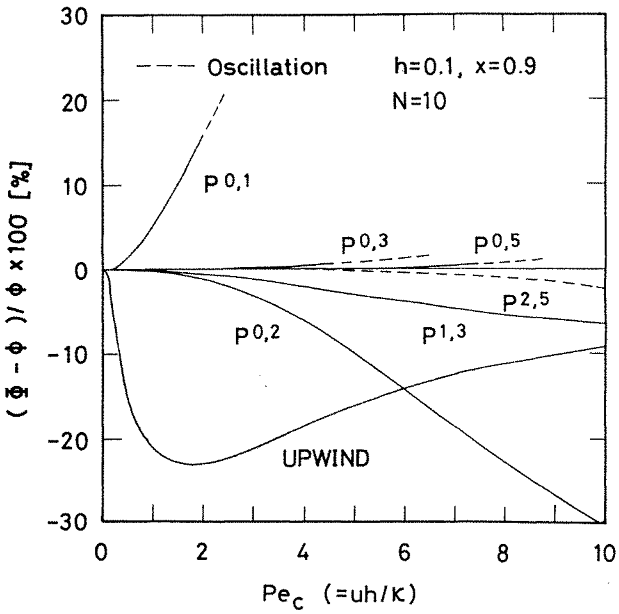

Figure 6. Relative error

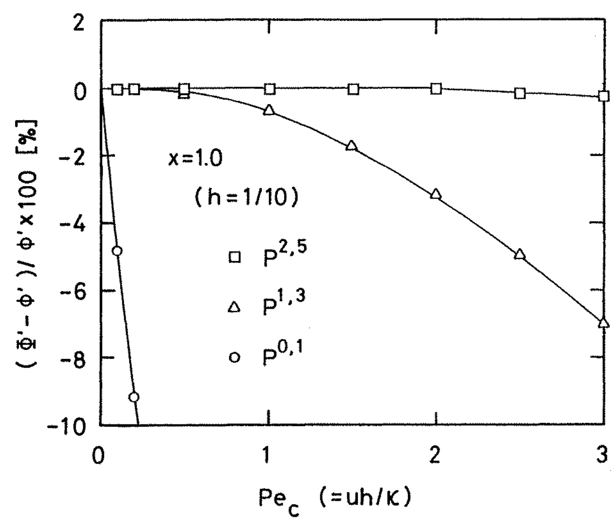

Figure 7. Relative error of derivative

10·5. However, as shown in Figure 8. The oscillations occur over every element for Pe_c larger than 1·68, and so these values do not have a physical meaning. The above results are listed in Table II.

We now discuss the errors. The behaviours of the maximum errors of ϕ_k for $P^{0,1}$, $P^{1,3}$ and $P^{2,5}$ are shown in Figure 9. For sufficiently small u the order of the convergence for h is almost constant, and even using the maximum norms at the nodes the order of the convergence is almost same as for the 2-norm under the basic elliptic condition.[8]

Table II. Upper limit for cell Peclet number ($Pe_c = uh/\kappa$) in stability and monotonicity criteria

	$P^{0,1}$	$P^{0,2}$	$P^{0,3}$	$P^{1,3}$	$P^{2,5}$
Standerd Galerkin					
based on stability external nodes	2·0	∞	4·644	∞	$\simeq 2.37$*
based on monotonicity at external and internal nodes	2·0	$\sqrt{12}$ (3.464)	4·464	$\simeq 10·5$	$\simeq 2·37$*
based on stability in entire domain (or Φ' stability) (or Φ' stability)	$\simeq 0·0$	$\simeq 0·0$	$\simeq 0·0$	$\simeq 1·68$*	$\simeq 2·03$
Φ'' stability at all nodes					$\simeq 1·95$
Elimination of internal nodal values or derivative values					
based on stability in entire domain	2·0	∞	4·644	∞	44·4
based on consistency in entire domain	2·0	3·464	4·644	4·367	6·118
Christie and Mitchell[7]					
based on stability at external nodes	2·0	∞	(4·8)		
Jensen and Finlayson[2]					
(A) based on monotonicity at nodes on each element	2·0	4·0	4·644	∞	
(B) based on stability in each element	2·0	2·0	4·644		

*Experimental result (oscillation on 9 up to 11 decimals with 4 times-precision calculaion).

Table III. Accuracy on $P^{0,1}$, $P^{1,3}$, $P^{2,5}$

Element	$P^{0,1}$						$P^{1,3}$						$P^{2,5}$					
u	0·01,	0·1,	1,	5,	10,	19	0·01,	0·1,	1,	5,	10,	20	0·01,	0·1,	1,	5,	10,	20
$\max\lvert\Phi - \Phi\rvert_{node}$	2·0	2·0	2·0	2·0	2·0	2·0	4·0	4·0	3·9	3·7	3·6	3·3	5·9	5·9	5·9	5·8	5·5	5·1
$\max\lvert\Phi' - \Phi'\rvert_{node^*}$	1·0	1·0	0·95	0·95	0·85	0·68	3·0	2·9	2·7	3·2	3·2	2·5	5·0	5·0	5·0	5·0	4·8	4·6
$\max\lvert\Phi' - \Phi'\rvert_{\text{center of element}}$	2·0	2·0	2·0	1·9	1·6	1·3												
$\max\lvert\Phi'' - \Phi''\rvert_{node}$													4·0	4·0	4·0	4·0	3·9	3·5

Theoretical values under the assumption of elliptic condition.

	$P^{0,1}$						$P^{1,3}$						$P^{2,5}$					
$\lVert\Phi - \Phi\rVert$			2						4						6			
$\lVert\Phi' - \Phi'\rVert$			1						3						5			
$\lVert\Phi'' - \Phi''\rVert$															4			

*Mean value of derivative values in adjacent elements for $P^{0,1}$.

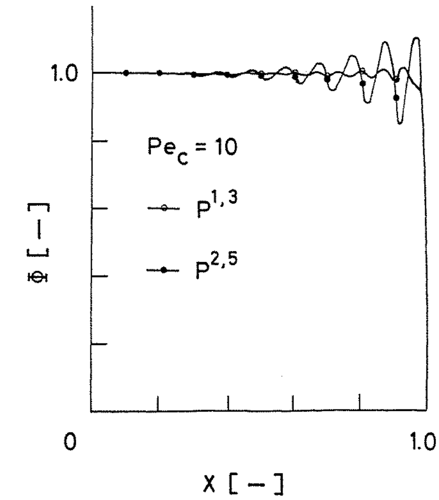

Figure 8. Solution of diffusion–convection equation with Hermitian polynomials $P^{1,3}, P^{2,5}$

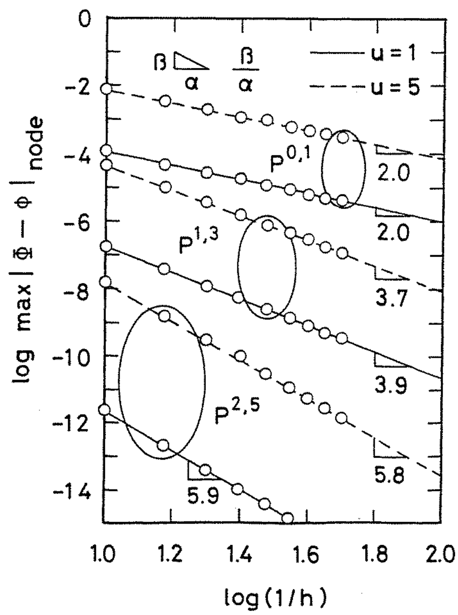

Figure 9. Accuracy on $P^{0,1}$, $P^{1,3}$, $P^{2,5}$

If we consider the nodal value, the accuracy of the FE solution for $P_{0,1}$ is almost independent of u, but the accuracies of the FE solutions for $P_{1,3}$ and $P_{2,5}$ decrease with increasing u. The accuracies of ϕ_k, ϕ_k' and ϕ_k'' are listed in Table III.

5. THE STABLE ALGORITHM

For the even-order Lagrangian family of finite elements, by eliminating the internal nodal value and by the use of a functional interpolation method a stable solution is obtained. In the case

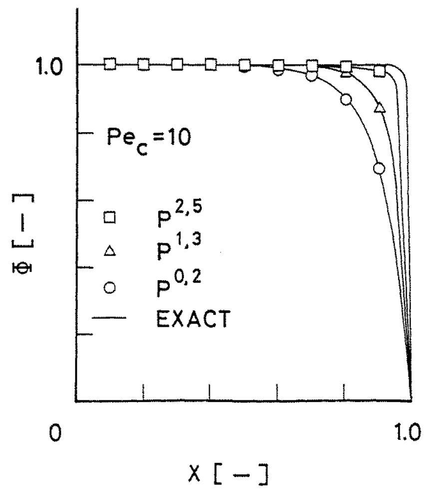

Figure 10. Exponentially interpolated solution

of the Hermitian family of finite elements by eliminating the derivative at each node the stable domain can be extended. For $P^{1,3}$ the stable calculation can be performed for every Pe_c in the same way as for the Lagrange even-order elements. On the other hand for $P^{2,5}$ the stable calculation can be performed up to $Pe_c = 44{\cdot}4$. These results are shown in Figure 10.

6. CONCLUSION

The stability and accuracy of the Hermite-type FEM for steady diffusion–convection problems have been theoretically and numerically discussed. We clarified the bound and the effectiveness of the Hermitian family of finite element based on the fact that the characteristic solution of the FE equation is approximated by a rational function of Pe_c.

Summarizing our results, they are as follows:

1. By the use of the first kind third order Hermitian family of finite elements: $P^{1,3}$, there is obtained a non-oscillating stable FE solution for the nodal value.
2. For the second kind 5th order Hermitian family of finite elements: $P^{2,5}$ the stable solution is not obtained.
3. For large Pe_c the derivative does not have a physical meaning because oscillation of the solution over an element occurs. For Pe_c less than 1·68 the solution for $P^{1,3}$ and for Pe_c less than 2·03 the solution for $P^{2,5}$ does not oscillate, so that solutions with high accuracy are obtained.
4. For sufficiently small u the accuracy of the FEM solution with Hermitian polynomials has order $n+1$, but its accuracy decreases as u increases.
5. For the proposed technique, which is eliminating the derivative at each node and using the exponential function's interpolation method over each element, the stable solutions are obtained over the domain independently of Pe_c and for $P^{2,5}$ the stable solutions are obtained for Pe_c less than 44·4.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. Masatoshi Ikeuchi of Okayama University of Science for his many valuable suggestions and comments on this work. The first author is also indebted to

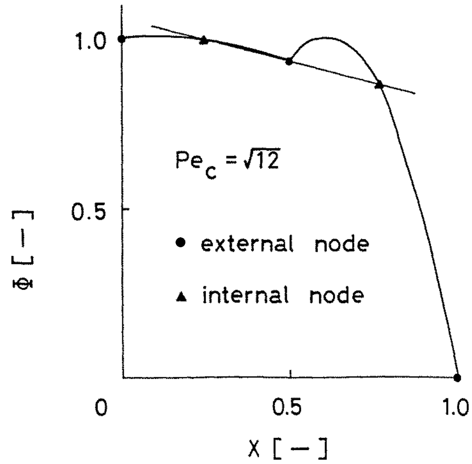

Figure 11. Solution of diffusion–convection equation with quadratic Lagrangian polynomial. (Case of two elements)

Professor Mutsuto Kawahara of the Civil Engineering Department of Chuo University Tokyo for encouraging advice on this work.

APPENDIX. THE CONSISTENT DOMAIN[4,6]

In the case of the odd-order Lagrangian family of finite elements we defined the consistent domain as extending from 0 until the singular point of $\Lambda_1 = b_n(Pe_c)/a_n(Pe_c)$. On the other hand for the even-order element it is defined by

$$Pe_c < f(c) = 2{\cdot}0 + 1{\cdot}4(n-1)$$

where n is an even number. For the second order element ($\sqrt{12}$ in correct), as shown in Figure 11, the gradient of among every elements (including internal points) has monotonicity as following relation:

$$|\phi_i - \phi_{i-1}| < |\phi_{i+1} - \phi_i|$$

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